OPE-Algebras and their Modules

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Abstract

Vertex algebras formalize the subalgebra of holomorphic fields of a conformal field theory. OPE-algebras were proposed as a generalization of vertex algebras that formalizes the algebra of all fields of a conformal field theory. We prove some basic results about them: The state-field correspondence is an OPE-algebra isomorphism and Dong's lemma and the existence theorem hold for multiply local OPE-algebras; locality implies skew-symmetry; if skew-symmetry holds then duality implies locality for modules and they are equivalent for algebras. We define modules over OPE-algebras.

1. Introduction

Conformal field theory (CFT) and vertex algebras made possible dramatic progress in moonshine, representation theory, quantum topology, moduli spaces, orbifolds, mirror symmetry, and the geometric Langlands program. Recently, Kapustin and Orlov [KO03] proposed a mathematical definition of CFTs generalizing vertex algebras. We call their notion OPE-algebras. "OPE" stands for operator product expansion.

Kapustin and Orlov's paper is about mirror symmetry. They construct an OPE-algebra $V(T)$ from a (complex) torus T with a flat metric and a constant 2-form (a B-field). This is the sigma model of T. The lack of injectivity of the correspondence $T \mapsto V(T)$ is called T-duality. In the context of $N = 2$ superconformal OPE-algebras, Kapustin and Orlov determine under which conditions $V(T_1), V(T_2)$ are isomorphic and when they are mirror to each other.

OPE-algebras may be defined in terms of a space of fields or in terms of a state-field correspondence, see section 2. The former approach is closer to the Wightman axioms and may apply to massive deformations of CFTs as well.

According to Polyakov and Kadanoff, the space of fields of a CFT endowed with the OPE-coefficients as products is an algebra with infinitely many multiplications. In section 3 we define infinitely many products of two local fields using the OPE. In section 4 we prove Dong's lemma for multiple locality and use it to prove the existence theorem and the fact that the state-field correspondence Y is an OPE-algebra isomorphism. Thus Y is the adjoint representation. The existence theorem provides an OPE-algebra structure on a vector space V once a generating set of multiply local fields on V is given. For vertex algebras, these results are due to Li [Li96], Lian, Zuckerman [LZ94], Kac [Kac98], Frenkel, Kac, Radul, Wang [FKRW95], and Meurman, Primc [MP99].

OPE-algebras are defined in terms of locality. Borcherds [Bor86] originally defined vertex algebras in terms of the associativity formula and skew-symmetry. Li [Li96] proved the equivalence of these two formulations for vertex algebras. The associativity formula is the statement that the state-field correspondence Y is a vertex algebra morphism. It implies duality. Geometrically, the map Y corresponds to a 3-punctured sphere and locality and duality are two identities between the three

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ways of cutting a 4-punctured sphere into two 3-punctured spheres [Hua97]. Vertex algebra modules can be equivalently defined in terms of the associativity formula or duality or the Jacobi identity.

In section 5 we define skew-symmetry for OPE-algebras and prove that locality implies skew-symmetry. In section 6 we define duality and prove for algebras that if skew-symmetry holds then duality is equivalent to locality. In section 7 we prove for modules that skew-symmetry and duality imply locality. In section 8 we define the notion of a module over an OPE-algebra V and prove that if V is uniformly local then V is a V-module. For vertex algebras, these results are due to Li [Li96]. As in the case of vertex algebras, OPE-algebra modules are interesting for example because of their relation to modular invariance.

It is easy to see that any vertex algebra is an OPE-algebra. Kapustin and Orlov prove the non-trivial result that the subspace of holomorphic states of an OPE-algebra is a vertex algebra.

Right now, the OPE-algebras $V(T)$ are the only examples of OPE-algebras besides vertex algebras. These examples are very special and belong to the class of *additive* OPE-algebras, see [Ros]. Additive OPE-algebras are multiply and uniformly local and can be defined in terms of a (5-term) Jacobi identity. Further examples of OPE-algebras should exist, e.g. orbifolds of $V(T)$ and various rational models (minimal models, WZW-models etc.). For their construction one should use results about intertwiners, e.g. the fundamental results of Huang, see [Hua].

Conventions. We work over a field $\mathbb{K}$ of characteristic 0. We always denote by E a vector space and by A an associative algebra. For $a \in A$ and $n \in \mathbb{K}$, define $a^{(n)} := a^n/n!$ if $n \in \mathbb{N} := \mathbb{Z}_{\geqslant 0}$ and $a^{(n)} := 0$ otherwise. Let $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$.

We work with super objects without making this explicit in our terminology, e.g. super vector spaces are just called vector spaces. Linear maps need not be even, i.e. preserve the super grading. Supersigns are written as powers of $\zeta := -1$. The parity of a is denoted by $\tilde{a} \in \{\bar{0}, \bar{1}\}$.

2. OPE-Algebras

We define OPE-algebras in terms of a space of fields and show that this definition is equivalent to a definition in terms of a state-field correspondence.

An E-valued *distribution* is a formal sum $a(z) = \sum_{n\in\mathbb{K}} a_n z^{-n-1}$ where z is a formal variable and $a_n \in E$. Here and in the following we only discuss the case of one variable.

The vector space $E\{z\}$ of distributions is a module over the group ring $\mathbb{K}[z^{\mathbb{K}}]$ of $\mathbb{K}$, $z^m a(z) := \sum a_{n+m} z^{-n-1}$. Let $E\langle z\rangle$ be the submodule generated by the subspace of power series $E[\![z]\!] := \{\sum_{n\in\mathbb{Z}_{<0}} a_n z^{-n-1}\}$. There exists a morphism $E\langle z, w\rangle \to E\langle z\rangle, a(z, w) \mapsto a(z, z)$. A linear map $E \otimes F \to G$ induces morphisms $E\{z\} \otimes F\{w\} \to G\{z, w\}$ and $E\langle z\rangle \otimes F\langle z\rangle \to G\langle z\rangle$. Define $\partial_z a(z) := -\sum n\, a_{n-1} z^{-n-1}$.

Let $z, \bar{z}$ be variables. We shall use the following notations time and again. Define $a(\check{z}) := a(z, \bar{z})$ and $E\{\check{z}\} := E\{z, \bar{z}\}$. If S, T are sets and $\check{s} \in S \times T$ then $s, \bar{s}$ denote the first and the second component of $\check{s}$. If $a_s(z), \bar{a}_{\bar{s}}(z) \in A\{z\}$ then we define $\check{a}_{\check{s}}(\check{z}) := a_s(z)\bar{a}_{\bar{s}}(\bar{z})$. For example, if $\check{n} \in \mathbb{K}^2$ and $\check{a} \in A^2$ then $\check{z}^{\check{n}} = z^n \bar{z}^{\bar{n}}$ and $\check{a}^{(\check{n})} = a^{(n)}\bar{a}^{(\bar{n})}$. Thus $a(\check{z}) = \sum_{\check{n}\in\mathbb{K}^2} a_{\check{n}} \check{z}^{-\check{n}-1}$ with $1 := (1, 1) \in \mathbb{K}^2$.

Unless stated otherwise, from now on all distributions will be $\mathrm{End}(E)$-valued.

A distribution $a(\check{z})$ is a *field* if $a(\check{z})b \in E\langle\check{z}\rangle$ for any $b \in E$. Let $\mathcal{F}_r(E)$ be the space of fields $a(\check{z}_1, \dots, \check{z}_r)$ and $\mathcal{F}(E) := \mathcal{F}_1(E)$.

Let $1 \in E_{\bar{0}}$ and $\check{T} \in \mathrm{End}(E)^2_{\bar{0}}$. We call 1 *invariant* if $T1 = \bar{T}1 = 0$. Define $s_1 : \mathrm{End}(E)\{\check{z}\} \to E$ by $a(\check{z}) \mapsto a_{-1}(1)$. A distribution $a(\check{z})$ is *weakly creative* for 1 if $a(\check{z})1 \in E[\![\check{z}]\!]$. It is *creative* for 1 and $\check{T}$ if $a(\check{z})1 = e^{\check{z}\check{T}} s_1 a(\check{z})$. It is *translation covariant* for $\check{T}$ if $[T, a(\check{z})] = \partial_z a(\check{z})$ and $[\bar{T}, a(\check{z})] = \partial_{\bar{z}} a(\check{z})$. A subspace $\mathcal{F} \subset \mathrm{End}(E)\{\check{z}\}$ is *complete* if $s_1\mathcal{F} = E$.

If 1 is invariant and $a(\check{z})$ translation covariant and weakly creative then $a(\check{z})$ is creative. If $[T,\bar{T}]=0$ then translation covariance is equivalent to $e^{\check{w}\check{T}}a(\check{z})e^{-\check{w}\check{T}}=a(\check{z}+\check{w})$ where $a(z+w):=e^{w\partial_z}a(z)$.

For $\check{n}\in\mathbb{K}^2$, define

$$(\check{z}-\check{w})^{\check{n}} \;:=\; \sum_{\check{i}\in\mathbb{N}^2}(-1)^{\check{i}}\binom{\check{n}}{\check{i}}\,\check{z}^{\check{n}-\check{i}}\check{w}^{\check{i}}.$$

For $\check{n}\in\check{\mathbb{K}}:=\{\check{n}\in\mathbb{K}^2\mid n-\bar{n}\in\mathbb{Z}\}$, define $(-1)^{\check{n}}:=(-1)^{n-\bar{n}}$ and $(\check{z}-\check{w})^{\check{n}}_{w>z}:=(-1)^{\check{n}}(\check{w}-\check{z})^{\check{n}}$.

Distributions $a(\check{z}),b(\check{z})$ are *local* if there exist $c^i(\check{z},\check{w})\in\mathcal{F}_2(E)$ and $\check{h}_i\in\check{\mathbb{K}}$ such that

$$a(\check{z})b(\check{w}) \;=\; \sum_{i=1}^{r}\frac{c^i(\check{z},\check{w})}{(\check{z}-\check{w})^{\check{h}_i}},\qquad \zeta^{ab}\,b(\check{w})a(\check{z}) \;=\; \sum_{i=1}^{r}\frac{c^i(\check{z},\check{w})}{(\check{z}-\check{w})^{\check{h}_i}_{w>z}}. \tag{1}$$

Equations (1) are called *OPEs* in $z>w$ and $w>z$.

If $\mathcal{P}$ is a property of elements or pairs of elements of a set S then we say that a subset $T\subset S$ satisfies $\mathcal{P}$ if $\mathcal{P}$ is satisfied for any element, resp., any pair of elements of T.

DEFINITION. A vector space V together with a vector $1\in V_{\bar{0}}$ and a subspace $\mathcal{F}\subset\operatorname{End}(V)\{\check{z}\}$ is an *OPE-algebra* if there exists $\check{T}\in\operatorname{End}(V)^2_{\bar{0}}$ such that 1 is invariant and $\mathcal{F}$ is weakly creative, translation covariant, complete, and local.

The following result is proven in [KO03].

GODDARD'S UNIQUENESS THEOREM. *Let $\mathcal{F}\subset\operatorname{End}(E)\{\check{z}\}$ be a creative, complete, local subspace. Then $s_1:\mathcal{F}\to E$ is an isomorphism.* $\square$

Let V be an OPE-algebra. The theorem shows that the inverse $Y:=(s_1|_{\mathcal{F}})^{-1}:V\to\operatorname{End}(V)\{\check{z}\}$ exists and if $a(\check{z})\in\operatorname{End}(V)\{\check{z}\}$ is creative and local to $\mathcal{F}$ then $a(\check{z})=Y(s_1a(\check{z}))\in\mathcal{F}$.

Let V be a vector space. To give an even linear map $Y:V\to\operatorname{End}(V)\{\check{z}\}, a\mapsto a(\check{z})=\sum a_{(\check{n})}\check{z}^{-\check{n}-1}$, is equivalent to giving an even multiplication $V\otimes V\to V, a\otimes b\mapsto a_{(\check{n})}b$, for any $\check{n}\in\mathbb{K}^2$. We call this a $\mathbb{K}^2$*-fold algebra*.

Let V be a $\mathbb{K}^2$-fold algebra, $1\in V_{\bar{0}}$, and $\check{T}\in\operatorname{End}(V)^2_{\bar{0}}$. Define $\mathcal{F}_Y:=Y(V)$. Define $\check{T}_1\in\operatorname{End}(V)^2_{\bar{0}}$ by $T_1a:=a_{(-2,-1)}1$ and $\bar{T}_1a:=a_{(-1,-2)}1$. We call 1 a *(weak) right identity* if $\mathcal{F}_Y$ is (weakly) creative for 1 and $\check{T}_1$ and $s_1a(\check{z})=a$ for any $a\in V$.

We call 1 a *left identity* if $Y(1)=1(z)$ where $1(z):=\mathrm{id}_V$ is the *identity field*. An *identity* is a left and right identity. Assume that $[T,\bar{T}]=0$. The pair $\check{T}$ is a *translation generator* if $\mathcal{F}_Y$ is translation covariant for $\check{T}$. It is a *translation endomorphism* if $(Ta)(\check{z})=\partial_z a(\check{z})$ and $(\bar{T}a)(\check{z})=\partial_{\bar{z}}a(\check{z})$.

To give a vector space V, a vector $1\in V_{\bar{0}}$, and a weakly creative subspace $\mathcal{F}\subset\operatorname{End}(V)\{\check{z}\}$ such that $s_1:\mathcal{F}\to V$ is an isomorphism is equivalent to giving a $\mathbb{K}^2$-fold algebra with a weak right identity. Moreover, Proposition 3 shows that OPE-algebras are unital and thus a weak right identity is unique. This shows that the above definition is equivalent to the following one.

DEFINITION. An *OPE-algebra* is a $\mathbb{K}^2$-fold algebra such that there exist a translation generator $\check{T}$ and an invariant weak right identity 1 and $\mathcal{F}_Y$ is local.

It follows that $\check{T}=\check{T}_1$ and that $\check{T}$ is a translation endomorphism. Moreover, one need not require that $[T,\bar{T}]=0$, it is a consequence. Proposition 3 shows that 1 is an identity. A *morphism* of OPE-algebras is a morphism of the underlying unital $\mathbb{K}^2$-fold algebras.

If V,W are OPE-algebras then so is $V\otimes_{\mathbb{K}}W$.

3. The Algebra of Fields

We define a field $a(\check{z})_{(\check{n})}b(\check{z})$ for any local distributions $a(\check{z}), b(\check{z})$ and any $\check{n} \in \mathbb{K}^2$ and prove some basic properties of this (partial) $\mathbb{K}^2$-fold algebra.

Distributions in $E[\![z^{\pm 1}]\!] := \{\sum_{n\in\mathbb{Z}} a_n z^{-n-1}\}$ are called *holomorphic*. Define $\mathcal{F}_z(E) := \mathcal{F}(E) \cap \mathrm{End}(E)[\![z^{\pm 1}]\!]$. Define $(z-w)^n[a(z), b(\check{w})] := (z-w)^n a(z)b(\check{w}) - \zeta^{ab}(z-w)^n_{w>z} b(\check{w})a(z)$ for $a(z) \in \mathcal{F}_z(E), b(\check{z}) \in \mathcal{F}(E)$, and $n \in \mathbb{Z}$. The following two results are proven in [KO03].

PROPOSITION 1. i) *Let $c^i(\check{z}, \check{w}) \in E\langle \check{z}, \check{w}\rangle$ and $\check{h}_i \in \mathbb{K}^2$ such that $\check{h}_i \notin \check{h}_j + \mathbb{Z}^2$ for $i \neq j$ and*

$$\sum_{i=1}^{r} c^i(\check{z}, \check{w})\,(\check{z} - \check{w})^{\check{h}_i} \;=\; 0.$$

Then $c^i(\check{z}, \check{w}) = 0$ for any i.

ii) *Let $a(z) \in \mathcal{F}_z(E)$ and $b(\check{z}) \in \mathcal{F}(E)$. Then $a(z), b(\check{z})$ are local iff there exists $N \in \mathbb{Z}$ such that $(z-w)^N[a(z), b(\check{w})] = 0$. In this case we have the OPEs*

$$a(z)b(\check{w}) \;=\; \frac{c(z, \check{w})}{(z-w)^N}, \qquad \zeta^{ab} b(\check{w})a(z) \;=\; \frac{c(z, \check{w})}{(z-w)^N_{w>z}}.$$

□

An OPE (1) is *reduced* if $\check{h}_i \notin \check{h}_j + \mathbb{Z}^2$ for $i \neq j$. Reduced OPEs always exist for local distributions.

Let $a(\check{z}), b(\check{z}) \in \mathrm{End}(E)\{\check{z}\}$ be local with OPE (1). For $\check{n} \in \mathbb{K}^2$, define

$$a(\check{w})_{(\check{n})}b(\check{w}) \;:=\; \sum_{i=1}^{r} \partial_{\check{z}}^{(\check{h}_i - 1 - \check{n})} c^i(\check{z}, \check{w})|_{\check{z}=\check{w}}.$$

Proposition 1 i) and $\partial_{\check{z}}^{(\check{n}+\check{i})}((\check{z}-\check{w})^{\check{i}} c(\check{z}, \check{w}))|_{\check{z}=\check{w}} = \partial_{\check{z}}^{(\check{n})} c(\check{z}, \check{w})|_{\check{z}=\check{w}}, \check{i} \in \mathbb{N}^2$, show that this definition does not depend on the choice of the OPE. The fields $a(\check{z})_{(\check{n})}b(\check{z})$ are the Taylor coefficients of a non-existing Taylor expansion of $c^i(\check{z}, \check{w})$. If (1) is reduced then $c^i(\check{z}, \check{z}) = a(\check{z})_{(\check{h}_i - 1)}b(\check{z})$.

If $a(z) \in \mathcal{F}_z(E)$ and $b(\check{z}) \in \mathcal{F}(E)$ then

$$a(w)_{(\check{n})}b(\check{w}) \;=\; a(w)_{(n)}b(\check{w}) \;:=\; \mathrm{res}_z (z-w)^n [a(z), b(\check{w})]$$

for $\check{n} \in \mathbb{Z} \times \{-1\}$ and $a(z)_{(\check{n})}b(\check{z}) = 0$ otherwise, see [KO03]. Thus the products $a(\check{z})_{(\check{n})}b(\check{z})$ generalize the products $a(z)_{(n)}b(\check{z})$ known from the theory of vertex algebras [Li96, LZ94, Kac98]. Note that $a(z)_{(n)}b(\check{z})$ is always defined whereas $a(\check{z})_{(\check{n})}b(\check{z})$ is only defined if an OPE in $z > w$ exists.

Let V be an OPE-algebra. It is shown in [KO03] that $V_z := \{a \in V \mid Ya \in \mathrm{End}(V)[\![z^{\pm 1}]\!]\}$ is a *vertex subalgebra*, i.e. $V_z \subset V$ is a unital $\mathbb{K}^2$-fold subalgebra, $a_{(\check{n})}b = 0$ for $a, b \in V_z, \check{n} \notin \mathbb{Z} \times \{-1\}$, and $(V, (_{(n,-1)})_{n\in\mathbb{Z}})$ is a vertex algebra. This subalgebra is the *chiral algebra* of V. Moreover, they show that $(a_{(n,-1)}b)(\check{z}) = a(z)_{(n)}b(\check{z})$ for any $a \in V_z, b \in V$. Similarly, the *anti-chiral* algebra $V_{\bar{z}}$ is defined and $V_z, V_{\bar{z}}$ commute, i.e. $[a(z), b(\bar{z})] = 0$.

For $a(z) \in E\{z\}$ and $S \subset \mathbb{K}$, define $\mathrm{supp}_z a(z) := \{n \mid a_n \neq 0\}$ and $a(z)|_S := \sum_{n\in S} a_n z^{-n-1}$. Let $E\{\check{z}\}' := \{a(\check{z}) \mid \mathrm{supp}_{\check{z}} a(\check{z}) \subset \check{\mathbb{K}}\}$. A $\mathbb{K}^2$-fold algebra V is a *bounded* $\check{\mathbb{K}}$*-fold* algebra if $\mathcal{F}_Y \subset \mathcal{F}(V) \cap \mathrm{End}(V)\{\check{z}\}'$.

PROPOSITION 2. *Let $a(\check{z}), b(\check{z}) \in \mathrm{End}(E)\{\check{z}\}$ be local.*

i) *$(\partial_z, \partial_{\bar{z}})$ is a translation endomorphism and a translation generator for $a(\check{z}), b(\check{z})$ (i.e. we have $\partial_z a(\check{z})_{(\check{n})}b(\check{z}) = -n\, a(\check{z})_{(n-1,\bar{n})}b(\check{z})$ etc.). The identity field $1(z)$ is an identity for any $c(\check{z}) \in \mathcal{F}(E)$ (i.e. we have $1(z)_{(\check{n})}c(\check{z}) = \delta_{\check{n},-1}c(\check{z})$ etc.). Moreover, $\check{T}_{1(z)} = (\partial_z, \partial_{\bar{z}})$.*

ii) *If $a(\check{z}), b(\check{z})$ are creative and translation covariant then so is $a(\check{z})_{(\check{n})}b(\check{z})$ and $s_1(a(\check{z})_{(\check{n})}b(\check{z})) = a_{\check{n}}s_1 b(\check{z})$ for any $\check{n}$.*

iii) *OPE-algebras are bounded $\check{\mathbb{K}}$-fold algebras and $c^i(\check{z}, \check{w}) \in \operatorname{End}(E)\{\check{z}, \check{w}\}'$.*

Proof. Part i) and the fact that $a(\check{z})_{(\check{n})}b(\check{z})$ is translation covariant follow from a direct calculation.

Let (1) be a reduced OPE of $a(\check{z}), b(\check{z})$ and $c \in E$. Define $S := \operatorname{supp}_{\check{z}} b(\check{z})c + \mathbb{Z}^2$ and $T := \mathbb{K}^2 \times (\mathbb{K}^2 \setminus S)$. Because $(\check{z}-\check{w})^{\check{h}} \in \mathbb{K}\{\check{z}\}[\![\check{w}]\!]$ we have

$$\sum_i \frac{c^i(\check{z},\check{w})c|_T}{(\check{z}-\check{w})^{\check{h}_i}} \;=\; \sum_i \frac{c^i(\check{z},\check{w})c}{(\check{z}-\check{w})^{\check{h}_i}}\bigg|_T \;=\; a(\check{z})b(\check{w})c|_T \;=\; 0.$$

Proposition 1 i) yields $c^i(\check{z},\check{w})c|_T = 0$. Hence $\operatorname{supp}_{\check{w}} c^i(\check{z},\check{w})c \subset S$. Similarly, the second OPE yields $\operatorname{supp}_{\check{z}} c^i(\check{z},\check{w})c \subset \operatorname{supp}_{\check{z}} a(\check{z})c + \mathbb{Z}^2$. In particular, $c^i(\check{z},\check{w})1 \in E[\![\check{z}^{\pm 1}, \check{w}^{\pm 1}]\!]$.

Define $S_i := (\check{h}_i + \mathbb{Z}^2) \times \mathbb{K}^2$. Because $(\check{z}-\check{w})^{\check{h}} \in \check{z}^{\check{h}}\mathbb{K}[\![\check{z}^{\pm 1}]\!]\{\check{w}\}$ we have

$$a(\check{z})b(\check{w})1|_{S_j} = \sum_i \frac{c^i(\check{z},\check{w})1}{(\check{z}-\check{w})^{\check{h}_i}}\bigg|_{S_j} = \frac{c^j(\check{z},\check{w})1}{(\check{z}-\check{w})^{\check{h}_j}}.$$

Since $b(\check{w})1 \in E[\![\check{w}]\!]$ we get $c^j(\check{z},\check{w})1 \in E\{\check{z}\}[\![\check{w}]\!]$. Similarly, the second OPE yields $c^i(\check{z},\check{w})1 \in E\{\check{w}\}[\![\check{z}]\!]$. Thus $c^i(\check{z},\check{w})1 \in E[\![\check{z},\check{w}]\!]$. This implies that $a(\check{z})_{(\check{n})}b(\check{z})$ is weakly creative. Applying the reduced OPE to 1 and setting $\check{w}=0$ yields

$$a(\check{z})s_1 b(\check{z}) \;=\; \sum_i \frac{c^i(\check{z},\check{w})1|_{\check{w}=0}}{\check{z}^{\check{h}_i}} \;\in\; \sum_i \check{z}^{-\check{h}_i}E[\![\check{z}]\!].$$

In particular, we obtain iii). Moreover, this shows that for any $\check{n} \in \mathbb{N}^2$ we have

$$a(\check{w})_{(\check{h}_i-1-\check{n})}b(\check{w})1|_{\check{w}=0} \;=\; \partial_{\check{z}}^{(\check{n})}c^i(\check{z},\check{w})1|_{\check{z}=\check{w}=0} \;=\; a_{\check{h}_i-1-\check{n}}s_1 b(\check{z}).$$

Hence $s_1(a(\check{z})_{(\check{n})}b(\check{z})) = a_{\check{n}}s_1 b(\check{z})$ for any $\check{n} \in \mathbb{K}^2$. Note that $c(\check{z})$ is creative iff $c(\check{z})$ is weakly creative and $s_1\partial_{\check{z}}^{(\check{n})}c(\check{z}) = \check{T}^{(\check{n})}s_1 c(\check{z})$ for any $\check{n}$. By i) we know that $\partial_{\check{z}}$ is a translation generator for local distributions. Thus

$$\begin{aligned} s_1\partial_{\check{z}}^{(\check{m})}(a(\check{z})_{(\check{n})}b(\check{z})) &= s_1\sum_{\check{i}=0}^{\check{m}}(-1)^{\check{i}}\binom{\check{n}}{\check{i}}a(\check{z})_{(\check{n}-\check{i})}b(\check{z}) + a(\check{z})_{(\check{n})}\partial_{\check{z}}^{(\check{m}-\check{i})}b(\check{z}) \\ &= \sum_{\check{i}=0}^{\check{m}}(-1)^{\check{i}}\binom{\check{n}}{\check{i}}a_{\check{n}-\check{i}}s_1 b(\check{z}) + a_{\check{n}}s_1\partial_{\check{z}}^{(\check{m}-\check{i})}b(\check{z}). \end{aligned}$$

On the other hand, since $a(\check{z})$ is translation covariant we have

$$\begin{aligned} \check{T}^{(\check{m})}s_1(a(\check{z})_{(\check{n})}b(\check{z})) \;&=\; \check{T}^{(\check{m})}a_{\check{n}}s_1 b(\check{z}) \\ &=\; \sum_{\check{i}=0}^{\check{m}}(-1)^{\check{i}}\binom{\check{n}}{\check{i}}a_{\check{n}-\check{i}}s_1 b(\check{z}) \;+\; a_{\check{n}}\check{T}^{(\check{m}-\check{i})}s_1 b(\check{z}). \end{aligned}$$

This shows that $a(\check{z})_{(\check{n})}b(\check{z})$ is creative. □

4. Multiple Locality

We prove Dong's lemma, the existence theorem, and that Y is an OPE-algebra isomorphism for multiply local fields.

For a permutation $\sigma\in\mathbb{S}_r$, $\check{n}\in\check{\mathbb{K}}$, and $i\neq j\in[r]$, define

$$(\check{z}_i-\check{z}_j)^{\check{n}}_{\sigma} \;:=\; \begin{cases} (\check{z}_i-\check{z}_j)^{\check{n}} & \text{if } \sigma^{-1}(i)<\sigma^{-1}(j) \text{ and}\\ (\check{z}_i-\check{z}_j)^{\check{n}}_{z_j>z_i} & \text{otherwise.}\end{cases}$$

Distributions $a^1(\check{z}),\dots,a^r(\check{z})\in\operatorname{End}(E)\{\check{z}\}$ are *multiply local* if there exist $r_{ij}\in\mathbb{N}, \check{h}^k_{ij}\in\check{\mathbb{K}}$, and $c^\alpha\in\mathcal{F}_r(E)$ for any $i,j\in[r]$, $k\in[r_{ij}]$, and $\alpha\in\prod_{i<j}[r_{ij}]$ such that for any $\sigma\in\mathbb{S}_r$ we have

$$\zeta'\, a^{\sigma 1}(\check{z}_{\sigma 1})\dots a^{\sigma r}(\check{z}_{\sigma r}) \;=\; \sum_\alpha \frac{c^\alpha(\check{z}_1,\dots,\check{z}_r)}{\prod_{i<j}(\check{z}_i-\check{z}_j)_\sigma^{\check{h}^{\alpha(i,j)}_{ij}}} \tag{2}$$

where ζ' is the obvious supersign. Equations (2) are also called *OPEs*. They are *reduced* if $\check{h}^k_{ij}\notin \check{h}^l_{ij}+\mathbb{Z}^2$ for $k\neq l$.

A pair of distributions is multiply local iff it is local. A subset $S\subset\operatorname{End}(E)\{\check{z}\}$ is *multiply local* if any finite family in S is multiply local. An OPE-algebra V is *multiply local* if $\mathcal{F}_Y$ is.

A subset $S\subset\operatorname{End}(E)\{\check{z}\}$ is *uniformly local* if there exist $r_{ab}\in\mathbb{N}$ and $\check{h}^k_{ab}\in\check{\mathbb{K}}$ for any $a(\check{z}),b(\check{z})\in S, k\in[r_{ab}]$ such that (2) is satisfied for any $a^i(\check{z})\in S$ with $\check{h}^k_{ij}:=\check{h}^k_{a^ia^j}$ and $c^\alpha 1\in E[\![\check{z}_1,\dots,\check{z}_r]\!]$. For example, additive OPE-algebras are uniformly local, see [Ros]. The proof of Proposition 2 shows that $c^\alpha 1\in E[\![\check{z}_1,\check{z}_2]\!]$ if $r=2$ and $a^1(\check{z}),a^2(\check{z})$ are local and weakly creative.

Dong's Lemma. i) *Let $a^1(\check{z}),\dots,a^r(\check{z})$ be multiply local and $a^1(\check{z}),a^2(\check{z})$ local. Then $a^1(\check{z})_{(\check{n})}$ $a^2(\check{z}),a^3(\check{z}),\dots,a^r(\check{z})$ are multiply local for any $\check{n}$.*

ii) *Let $S\subset\mathcal{F}(E)$ be multiply (uniformly) local. Then there exists a multiply (uniformly) local subspace $\langle S\rangle$ that is closed with respect to the products $a(\check{z})_{(\check{n})}b(\check{z})$, contains $1(z)$, and is generated by S as a unital $\mathbb{K}^2$-fold algebra.*

Proof. i) We may assume that (2) and $a^1(\check{z})a^2(\check{w})=\sum_{k=1}^{r_{12}}(\check{z}-\check{w})^{-\check{h}^k_{12}}\,c^k(\check{z},\check{w})$ are reduced OPEs. Let $\sigma\in\mathbb{S}_r$ such that $\sigma^{-1}(2)=\sigma^{-1}(1)+1$. Then

$$\zeta'\, a^{\sigma 1}(\check{z}_{\sigma 1})\dots a^{\sigma r}(\check{z}_{\sigma r}) \;=\; \sum_{k=1}^{r_{12}} \frac{a^{\sigma 1}(\check{z}_{\sigma 1})\dots c^k(\check{z}_1,\check{z}_2)\dots a^{\sigma r}(\check{z}_{\sigma r})}{(\check{z}_1-\check{z}_2)^{\check{h}^k_{12}}}.$$

Multiple locality and Proposition 1 i) yield

$$\zeta'\, a^{\sigma 1}(\check{z}_{\sigma 1})\dots c^k(\check{z}_1,\check{z}_2)\dots a^{\sigma r}(\check{z}_{\sigma r}) \;=\; \sum_{\alpha:\,\alpha(1,2)=k} \frac{c^\alpha(\check{z}_1,\dots,\check{z}_r)}{\prod_{(i,j)\neq(1,2)}(\check{z}_i-\check{z}_j)_\sigma^{\check{h}^{\alpha(i,j)}_{ij}}}$$

for any k. We have $a^1(\check{w})_{(\check{h}^k_{12}-1-\check{n})}a^2(\check{w})=\partial^{(\check{n})}_{\check{z}}c^k(\check{z},\check{w})|_{\check{z}=\check{w}}$ for any $\check{n}\in\mathbb{N}^2$. Thus by acting with $\partial^{(\check{n})}_{\check{z}_1}$ on the last equation and setting $\check{z}_1=\check{z}_2$ we obtain

$$\zeta'\, a^{\sigma 1}(\check{z}_{\sigma 1})\dots a^1(\check{z}_2)_{(\check{h}^k_{12}-1-\check{n})}a^2(\check{z}_2)\dots a^{\sigma r}(\check{z}_{\sigma r}) \;=\; \sum_{\alpha:\,\alpha(1,2)=k} \frac{\tilde{c}^\alpha(\check{z}_2,\dots,\check{z}_r)}{\prod_{1<i<j}(\check{z}_i-\check{z}_j)_\sigma^{\check{h}^{\alpha(i,j)}_{ij}+\delta_{i,2}(\check{h}^{\alpha(1,j)}_{1j}+\check{n})}}$$

for some fields $\tilde{c}^\alpha$.

ii) This follows from i) and its proof. $\square$

Theorem. *Let V be a multiply local OPE-algebra. Then $\mathcal{F}_Y$ is closed with respect to the products $a(\check{z})_{(\check{n})}b(\check{z})$ and $Y:V\to\mathcal{F}_Y$ is an OPE-algebra isomorphism.*

Proof. Dong's lemma and Proposition 2 ii) show that $\langle\mathcal{F}_Y\rangle$ is local and creative. Goddard's uniqueness theorem and Proposition 2 ii) imply that $a(\check{z})_{(\check{n})}b(\check{z})=Y(s_1(a(\check{z})_{(\check{n})}b(\check{z})))=(a_{(\check{n})}b)(\check{z})$. We have $Y1=1(z)$ since V is unital by Proposition 3. $\square$

Existence Theorem. *Let V be a vector space, $1 \in V_{\bar{0}}, \check{T} \in \mathrm{End}(V)^2_{\bar{0}}$, and $S \subset \mathcal{F}(V)$ a weakly creative, translation covariant, multiply (uniformly) local subset such that 1 is invariant and*

$$V \;=\; \mathrm{span}\{a^1_{\check{n}_1} \ldots a^r_{\check{n}_r} 1 \mid a^i(\check{z}) \in S,\, \check{n}_i \in \mathbb{K}^2,\, r \in \mathbb{N}\}.$$

Then there exists a unique multiply (uniformly) local OPE-algebra structure Y on V such that 1 is a weak right identity and $Y(s_1 a(\check{z})) = a(\check{z})$ for any $a(\check{z}) \in S$. We have $\mathcal{F}_Y = \langle S \rangle$.

Proof. Dong's lemma and Proposition 2 ii) show that $\langle S \rangle$ is creative, translation covariant, complete, and multiply (uniformly) local. Thus $(V, 1, \langle S \rangle)$ is an OPE-algebra satisfying the three properties of the theorem. If Y' is such an OPE-algebra structure then $\langle S \rangle = \mathcal{F}_{Y'}$ and $Y' = (s_1|_{\langle S \rangle})^{-1}$. □

5. Skew-Symmetry

We define skew-symmetry and prove that locality implies skew-symmetry.

Let V be a bounded $\check{\mathbb{K}}$-fold algebra and $\check{T} \in \mathrm{End}(V)^2_{\bar{0}}$. *Skew-symmetry* for $\check{T}$ is the identity

$$\zeta^{ab}\, b_{(\check{n})} a \;=\; \sum_{\check{i} \in \mathbb{N}^2} (-1)^{\check{n}+\check{i}}\, \check{T}^{(\check{i})}(a_{(\check{n}+\check{i})} b)$$

for any $\check{n} \in \check{\mathbb{K}}$. This is equivalent to $\zeta^{ab}\, b(\check{z})a = e^{\check{z}\check{T}} a(-\check{z})b$ where $c(-\check{z}) := \sum_{\check{n} \in \check{\mathbb{K}}} (-1)^{\check{n}} c_{\check{n}} \check{z}^{-\check{n}-1}$ for $c(\check{z}) \in E\{\check{z}\}'$.

Proposition 3. i) *Local distributions in* $\mathrm{End}(E)\{\check{z}\}$ *satisfy skew-symmetry for* $(\partial_z, \partial_{\bar{z}})$.

ii) *OPE-algebras satisfy skew-symmetry and are unital.*

Proof. i) Let $a(\check{z}), b(\check{z})$ be local with reduced OPE (1) and $\check{n} \in \mathbb{N}^2$. The OPE in $w > z$, the binomial formula, and the chain rule yield

$$\begin{aligned}
\zeta^{ab}\, b(\check{w})_{(\check{h}_i - 1 - \check{n})} a(\check{w}) &= (-1)^{\check{h}_i}\, \partial_{\check{z}}^{(\check{n})} c^i(\check{w}, \check{z})|_{\check{z}=\check{w}} \\
&= \sum_{\check{k} \in \mathbb{N}^2} (-1)^{\check{h}_i}\, (\partial_{\check{z}} + \partial_{\check{w}})^{(\check{k})} (-\partial_{\check{w}})^{(\check{n}-\check{k})} c^i(\check{w}, \check{z})|_{\check{z}=\check{w}} \\
&= \sum_{\check{k} \in \mathbb{N}^2} (-1)^{\check{h}_i + \check{n} + \check{k}}\, \partial_{\check{w}}^{(\check{k})} (a(\check{w})_{(\check{h}_i - 1 - \check{n} + \check{k})} b(\check{w})).
\end{aligned}$$

ii) Because of Proposition 2 ii), part i), and creativity of $a(\check{z})_{(\check{n}+\check{i})} b(\check{z})$ we have

$$\begin{aligned}
\zeta^{ab}\, b_{(\check{n})} a \;=\; \zeta^{ab}\, s_1(b(\check{z})_{(\check{n})} a(\check{z})) \;&=\; s_1 \sum_{\check{i} \in \mathbb{N}^2} (-1)^{\check{n}+\check{i}}\, \partial_{\check{z}}^{(\check{i})} (a(\check{z})_{(\check{n}+\check{i})} b(\check{z})) \\
&= \sum_{\check{i} \in \mathbb{N}^2} (-1)^{\check{n}+\check{i}}\, \check{T}^{(\check{i})} (a_{(\check{n}+\check{i})} b).
\end{aligned}$$

Skew-symmetry implies that 1 is a left identity iff 1 is a right identity. □

6. Duality and Locality

We define duality and prove for $\check{\mathbb{K}}$-fold algebras that if skew-symmetry holds then duality is equivalent to locality.

Let V be a bounded $\check{\mathbb{K}}$-fold algebra and M a bounded $\check{\mathbb{K}}$-fold V-*module*, i.e. M is a vector space with an even linear map $Y : V \to \mathcal{F}(M) \cap \mathrm{End}(M)\{\check{z}\}'$.

Elements $a \in V, c \in M$ are *dual* in the *direct channel* if there exist $d^j(\check{w},\check{x}) \in \mathcal{F}_2(V)$ and $\check{h}^j_{ac} \in \check{\mathbb{K}}$ such that

$$a(\check{x}+\check{w})b(\check{w})c \;=\; \sum_{j=1}^{s} \frac{d^j(\check{w},\check{x})b}{(\check{x}+\check{w})^{\check{h}^j_{ac}}}, \qquad (a(\check{x})b)(\check{w})c \;=\; \sum_{j=1}^{s} \frac{d^j(\check{w},\check{x})b}{(\check{w}+\check{x})^{\check{h}^j_{ac}}} \tag{3}$$

for any $b \in V$. Elements $a, b \in V$ are *local* on M in the *direct channel* if $a(\check{z}), b(\check{z}) \in \mathcal{F}(M)$ are local.

The term "direct channel" means that the singularity in (1) and (3) is the one corresponding to a, b, resp., a, c.

Proposition 4. *Let V be a bounded $\check{\mathbb{K}}$-fold algebra satisfying skew-symmetry for a translation generator $\check{T}$. Then $a, b \in V$ are local in the direct channel iff a, b are dual in the direct channel.*

The idea of the proof is to argue that $a(bc) = b(ac)$ iff $a(cb) = (ac)b$.

Proof. '$\Rightarrow$' Let (1) be an OPE of $a(\check{z}), b(\check{z})$. For $c \in V$, we have

$$\zeta^{\tilde{b}\tilde{c}}\, a(\check{x}+\check{w})c(\check{w})b \;=\; a(\check{x}+\check{w})e^{\check{w}\check{T}}b(-\check{w})c \;=\; e^{\check{w}\check{T}}a(\check{x})b(-\check{w})c \;=\; \sum_i \frac{e^{\check{w}\check{T}}c^i(\check{x},-\check{w})c}{(\check{x}+\check{w})^{\check{h}_i}}$$

and

$$\zeta^{\tilde{b}\tilde{c}}\,(a(\check{x})c)(\check{w})b \;=\; \zeta^{\tilde{a}\tilde{b}}\, e^{\check{w}\check{T}}b(-\check{w})a(\check{x})c \;=\; \sum_i \frac{e^{\check{w}\check{T}}c^i(\check{x},-\check{w})c}{(\check{w}+\check{x})^{\check{h}_i}}.$$

'$\Leftarrow$' This is proven in the same way. $\square$

7. Duality and Skew-Symmetry

We prove for modules that duality and skew-symmetry imply locality.

Let V be a bounded $\check{\mathbb{K}}$-fold algebra and M a bounded $\check{\mathbb{K}}$-fold V-module. Elements $a, b \in V$ are *dual* in the *exchange channel* if there exist $\check{h}^i_{ab} \in \check{\mathbb{K}}$ and for any $c \in M$ there exist $d^j(\check{w},\check{x}) \in \sum_{i=1}^r \check{x}^{-\check{h}^i_{ab}} M[\![\check{x}]\!]\langle\check{w}\rangle$ and $\check{h}^j_{ac} \in \check{\mathbb{K}}$ such that

$$a(\check{x}+\check{w})b(\check{w})c \;=\; \sum_{j=1}^{s} \frac{d^j(\check{w},\check{x})}{(\check{x}+\check{w})^{\check{h}^j_{ac}}}, \qquad (a(\check{x})b)(\check{w})c \;=\; \sum_{j=1}^{s} \frac{d^j(\check{w},\check{x})}{(\check{w}+\check{x})^{\check{h}^j_{ac}}}. \tag{4}$$

Proposition 5. *Let V be a bounded $\check{\mathbb{K}}$-fold algebra satisfying skew-symmetry for $\check{T} \in \mathrm{End}(V)^2_{\bar{0}}$ and M a bounded $\check{\mathbb{K}}$-fold V-module with translation endomorphism $\check{T}$. If $a, b \in V$ and b, a are both dual in the exchange channel then a, b are local in the direct channel.*

The idea of the proof is to argue that $a(bc) = (ab)c = (ba)c = b(ac)$.

Proof. Let $c \in M$. We may assume that equations (4) and

$$b(\check{x}+\check{z})a(\check{z})c \;=\; \sum_k \frac{e^k(\check{z},\check{x})}{(\check{x}+\check{z})^{\check{h}^k_{bc}}}, \qquad (b(\check{x})a)(\check{z})c \;=\; \sum_k \frac{e^k(\check{z},\check{x})}{(\check{z}+\check{x})^{\check{h}^k_{bc}}} \tag{5}$$

are satisfied where

$$d^j(\check{w},\check{x}) \;=\; \sum_{i,k} \check{w}^{-\check{h}^k_{bc}}\, \check{x}^{-\check{h}^i_{ab}}\, p^{ab}_{ijk}(\check{w},\check{x}), \tag{6}$$

$$e^k(\check{z},\check{x}) \;=\; \sum_{i,j} \check{z}^{-\check{h}^j_{ac}}\, \check{x}^{-\check{h}^i_{ab}}\, p^{ba}_{ijk}(\check{z},\check{x}), \tag{7}$$

$p^{ab}_{ijk}(\check{z},\check{w}), p^{ba}_{ijk}(\check{z},\check{w}) \in M[\![\check{z},\check{w}]\!]$, and $\check{h}^l_p \in \check{\mathbb{K}}$ such that $\check{h}^l_p \notin \check{h}^{l'}_p + \mathbb{Z}^2$ for $l \neq l'$ and $p \in \{ab, ac, bc\}$.

Inserting (6) into (4) we get

$$(a(\check{x})b)(\check{w})c \;=\; \sum_{i,j,k} (\check{w}+\check{x})^{-\check{h}^j_{ac}}\, \check{w}^{-\check{h}^k_{bc}}\, \check{x}^{-\check{h}^i_{ab}}\, p^{ab}_{ijk}(\check{w},\check{x}).$$

On the other hand, applying skew-symmetry and inserting (7) into (5) we get

$$\begin{aligned}(a(\check{x})b)(\check{w})c \;&= \zeta^{ab}\,(e^{\check{x}\check{T}} b(-\check{x})a)(\check{w})c\\ &= \zeta^{ab}\, e^{\check{x}\partial_{\check{w}}}(b(-\check{x})a)(\check{w})c\\ &= \zeta^{ab}\, e^{\check{x}\partial_{\check{w}}} \sum_{i,j,k} \check{w}^{-\check{h}^j_{ac}}\,(\check{w}-\check{x})^{-\check{h}^k_{bc}}\,(-\check{x})^{-\check{h}^i_{ab}}\, p^{ba}_{ijk}(\check{w},-\check{x})\\ &= \zeta^{ab} \sum_{i,j,k} (\check{w}+\check{x})^{-\check{h}^j_{ac}}\,\check{w}^{-\check{h}^k_{bc}}\,(-\check{x})^{-\check{h}^i_{ab}}\, p^{ba}_{ijk}(\check{w}+\check{x},-\check{x}).\end{aligned}$$

Thus Proposition 1 i) yields

$$p^{ab}_{ijk}(\check{w},\check{x}) \;=\; \zeta^{ab}\,(-1)^{\check{h}^i_{ab}}\, p^{ba}_{ijk}(\check{w}+\check{x},-\check{x}). \tag{8}$$

Inserting (6) into (4) we get

$$\begin{aligned}a(\check{z})b(\check{w})c \;&= e^{-\check{w}\partial_{\check{z}}} a(\check{z}+\check{w})b(\check{w})c\\ &= e^{-\check{w}\partial_{\check{z}}} \sum_{i,j,k} (\check{z}+\check{w})^{-\check{h}^j_{ac}}\,\check{w}^{-\check{h}^k_{bc}}\,\check{z}^{-\check{h}^i_{ab}}\, p^{ab}_{ijk}(\check{w},\check{z})\\ &= \sum_{i,j,k} \check{z}^{-\check{h}^j_{ac}}\,\check{w}^{-\check{h}^k_{bc}}\,(\check{z}-\check{w})^{-\check{h}^i_{ab}}\, p^{ab}_{ijk}(\check{w},\check{z}-\check{w}).\end{aligned}$$

In the same way, inserting (7) into (5) we obtain

$$b(\check{w})a(\check{z})c \;=\; \sum_{i,j,k} \check{z}^{-\check{h}^j_{ac}}\,\check{w}^{-\check{h}^k_{bc}}\,(\check{w}-\check{z})^{-\check{h}^i_{ab}}\, p^{ba}_{ijk}(\check{z},\check{w}-\check{z}).$$

Because p^{ab}_{ijk} and p^{ba}_{ijk} are power series (8) implies

$$p^{ab}_{ijk}(\check{w},\check{z}-\check{w}) \;=\; \zeta^{ab}\,(-1)^{\check{h}^i_{ab}}\, p^{ba}_{ijk}(\check{z},\check{w}-\check{z}).$$

Thus locality holds. □

8. Modules

We define the notion of a module over an OPE-algebra V and prove that if V is uniformly local then V is a V-module.

Let V be a bounded $\check{\mathbb{K}}$-fold algebra and M a bounded $\check{\mathbb{K}}$-fold V-module. Then M is *dual* if there exist $r_{ab} \in \mathbb{N}$ and $\check{h}^i_{ab} \in \check{\mathbb{K}}$ for any $a \in V, b \in V \cup M, i \in [r_{ab}]$ such that for any $a, b \in V, c \in M$ there exist $p_{ijk}(\check{w},\check{x}) \in M[\![\check{w},\check{x}]\!]$ for $i \in [r_{ab}], j \in [r_{ac}], k \in [r_{bc}]$ such that

$$a(\check{x}+\check{w})b(\check{w})c \;=\; \sum_{i,j,k} \frac{p_{ijk}(\check{w},\check{x})}{\check{x}^{\check{h}^i_{ab}}\,(\check{x}+\check{w})^{\check{h}^j_{ac}}\,\check{w}^{\check{h}^k_{bc}}}, \qquad (a(\check{x})b)(\check{w})c \;=\; \sum_{i,j,k} \frac{p_{ijk}(\check{w},\check{x})}{\check{x}^{\check{h}^i_{ab}}\,(\check{w}+\check{x})^{\check{h}^j_{ac}}\,\check{w}^{\check{h}^k_{bc}}}. \tag{9}$$

If M is dual then duality in the direct and in the exchange channel are satisfied.

Propositions 5 and 3 show that if V is an OPE-algebra, $\check{T}$ is a translation endomorphism of M, and M is dual then V is local on M. Thus the following definition makes sense.

Definition. Let V be an OPE-algebra. A *V-module* is a bounded $\check{\mathbb{K}}$-fold V-module M such that $\check{T}$ is a translation endomorphism of M, M is dual, and $Y : V \to \mathcal{F}(M)$ is a $\mathbb{K}^2$-fold algebra morphism.

Lemma. *Let $c^\alpha \in E\langle \check{z}_1, \dots, \check{z}_r\rangle$ and $\check{h}_{ij}^k \in \mathbb{K}^2$ for any $i, j \in [r], k \in [r_{ij}]$, and $\alpha \in \prod_{i<j}[r_{ij}]$ such that $\check{h}_{ij}^k \notin \check{h}_{ij}^l + \mathbb{Z}^2$ for $k \neq l$ and*

$$\sum_\alpha \frac{c^\alpha(\check{z}_1, \dots, \check{z}_r)}{\prod_{i<j}(\check{z}_i - \check{z}_j)^{\check{h}_{ij}^{\alpha(i,j)}}} \;=\; 0.$$

Then $c^\alpha = 0$ for any α.

Proof. From $(\check{z} - \check{w})^{\check{h}} \in \mathbb{K}[\check{w}^{\mathbb{K}}]\{\check{z}\}$ we get

$$\prod_{(i,j)\neq(i',j')} (\check{z}_i - \check{z}_j)^{\check{h}_{ij}^{\alpha(i,j)}} \;\in\; \mathbb{K}\langle \check{z}_1\rangle \dots \langle \check{z}_{i'}, \check{z}_{j'}\rangle\{\check{z}_{i'+1}\} \dots \widehat{\{\check{z}_{j'}\}} \dots \{\check{z}_r\}$$

for any $i' < j'$. Thus the claim follows from Proposition 1 i) by induction on the number of pairs $i < j$. □

Proposition 6. *Let V be a uniformly local OPE-algebra, $a, a^i \in V$, and (2) an OPE of $a^1(\check{z}), \dots, a^r(\check{z})$. Then*

$$c^\alpha(\check{z}_1, \dots, \check{z}_r)a \;\in\; \sum_\kappa \check{z}_1^{-\check{h}^{\kappa 1}_{a^1 a}} \dots \check{z}_r^{-\check{h}^{\kappa r}_{a^r a}} V[\![\check{z}_1, \dots, \check{z}_r]\!]$$

for any α where $\kappa \in \prod_{i=1}^r [r_{a^i a}]$.

Proof. We may assume that the OPE (2) is reduced. Consider multiple locality for $a^1(\check{z}_1), \dots, a^r(\check{z}_r)$, $a(\check{z})$ and $\sigma = 1$. Apply it to $1 \in V$ and set $\check{z} = 0$. By assumption, $c^{(\alpha,\kappa)}(\check{z}_1, \dots, \check{z}_r, 0)1 \in V[\![\check{z}_1, \dots, \check{z}_r]\!]$.

On the other hand, consider multiple locality for $a^1(\check{z}_1), \dots, a^r(\check{z}_r)$ and $\sigma = 1$. Apply it to a. In both cases we get $a^1(\check{z}_1) \dots a^r(\check{z}_r)a$ on the left-hand side. Equating the right-hand sides and applying the Lemma yields the claim. □

Corollary. *Let V be a uniformly local OPE-algebra. Then V is a V-module.*

Proof. The fact that V is dual follows from Proposition 6 and from Proposition 4 and its proof. That Y is a morphism follows from the theorem in section 4. □

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References

Bor86 Richard E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986), 3068–3071.

FKRW95 Edward Frenkel, Victor G. Kac, Andrey Radul, and Weiqiang Wang, *$W_{1+\infty}$ and $W(\mathfrak{gl}_N)$ with central charge N*, Comm. Math. Phys. **170** (1995), 337–357.

Hua Yi-Zhi Huang, *Differential equations and intertwining operators*, math.QA/0206206.

Hua97 Yi-Zhi Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progr. Math., vol. 148, Birkhäuser, 1997.

Kac98 Victor G. Kac, *Vertex algebras for beginners*, 2nd ed., University Lecture Series, vol. 10, Amer. Math. Soc., 1998.

KO03 Anton Kapustin and Dmitri Orlov, *Vertex algebras, mirror symmetry, and D-branes: the case of complex tori*, Comm. Math. Phys. **233** (2003), 79–136.

Li96 Haisheng Li, *Local systems of vertex operators, vertex superalgebras and modules*, J. Pure Appl. Algebra **109** (1996), 143–195.

LZ94 Bong H. Lian and Gregg J. Zuckerman, *Some classical and quantum algebras*, Lie theory and geometry (Jean-Luc Brylinski, Ranee Brylinski, Victor Guillemin, and Victor G. Kac, eds.), Progr. Math., vol. 123, Birkhäuser, 1994, pp. 509–529.

MP99 Arne Meurman and Mirko Primc, *Annihilating fields of standard modules of* $\mathfrak{sl}(2,\mathbf{C})^{\sim}$ *and combinatorial identities*, Mem. Amer. Math. Soc. **137** (1999), no. 652.

Ros Markus Rosellen, *Additive OPE-algebras*, in preparation.

Ros02 Markus Rosellen, *OPE-algebras*, Bonner Math. Schriften, vol. 352, Universität Bonn Mathematisches Institut, 2002, Dissertation. math.QA/0209025.

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